MECH 466

Automatic Control
Presentation Part 1

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http://www.researchcentre.apsc.ubc.ca/MECH466/

Laplace Transform Lectures

A set of lectures have been organized for the students who don't have a background in Laplace transform techniques. Please see the following web site for the dates, times, and the locations of these lectures:

http://www.researchcentre.apsc.ubc.ca/MECH466/

Each lecture will be repeated (given twice) for the benefit of those with time conflicts.

Tutorial Sessions

A set of tutorials have been organized to assist the students in their homework assignments. These tutorials will commence one week after announcing the first assignment. Please see the following web site for the dates, times, and the locations of these tutorials:

http://www.researchcentre.apsc.ubc.ca/MECH466/

Each tutorial session will be repeated (given twice) for the benefit of those with time conflicts.

MECH 466: AUTOMATIC CONTROL

4 Credits, First Semester 2006/07 (MWF 12:00-12:50 hours)
Room: West Mall Swing Space (SWNG) Room 122

<u>Instructor</u>
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<u>Prerequisite:</u> One of EECE 251, EECE 263, PHYS 209 [3-2*-0]

Course Objectives

This introductory course in control systems deals with modeling, response analysis, and control of dynamic systems, and the analysis and design of control systems. The objective of control is to make a ***dynamic system*** (***plant*** or ***process***) behave in a desired manner, according some ***performance specifications***. The system can be quite complex and may be subjected to known and unknown excitations (***inputs***), as in the case of an aircraft. The system may have many responses (***outputs***) as well. The device or means that generates the control signal (or, control command) according to some scheme (or, ***control law***) and that controls the response of the plant, is called the ***controller***. To determine a control action in ***feedback control***, the controller compares the measured response signals with their desired values. The plant and the controller are the two essential components of a ***control system***. A ***compensator*** (analog or digital; hardware or software) may be employed as well to improve the system performance or to enhance the controller. Modern control techniques are applicable in every branch of engineering, particularly in mechanical engineering and mechatronic systems. In ***time-domain*** techniques, the system is represented as a set of differential equations whose independent variable is time (t). In ***frequency-domain*** techniques, the system is represented as a set of input-output relations called transfer functions whose independent variable is frequency (ω). Topics covered in the course include analytical models, model linearization, response analysis, performance specification, stability analysis, root locus method, Bode plots, Nyquist criterion, system compensation, controller design, and digital computer control. Laboratory experiments will complement the lecture content.

<u>Textbook:</u>
MECHATRONICS by C.W. de Silva, CRC Press, Boca Raton, FL 2005 (a set of selected chapters as indicated in the Course Layout).

<u>Further Reading (Optional):</u>
Any standard book on Feedback Control. For example, *Feedback Control of Dynamic Systems* by Franklin, Powell, and Emami-Naeini **<u>or</u>** *Modern Control Systems* by Dorf and Bishop, both books published by Prentice Hall, Upper Saddle River, NJ.

MECH 466 -- COURSE LAYOUT

Week	Starts	Topics	Read
1	Sep 06	Introduction; Control Engineering	Sections 12.1
2	Sep 11	Performance Specification; Analytical Modeling	Sections 12.2, 2.1
3	Sep 18	Analogies; State Space Models	Sections 2.2, 2.3
4	Sep 25	DC Motor; Model Linearization	Sections 9.1, 9.2, 2.4
5	Oct 02	Transfer Functions; DC Motor Control	Section 2.11, 9.3 Appendix A
6	Oct 11	Time Response Analysis	Section 2.13 Appendix A
7	Oct 16	Control Methods; Error Constants; Sensitivity	Section 12.3
8	Oct 23	Control Methods; Error Constants; Sensitivity	Section 12.3
9	Oct 30	Stability Analysis; Routh-Hurwitz Criterion	Section 12.4

(Mid-term examination will be held on Monday, October 30, in class)

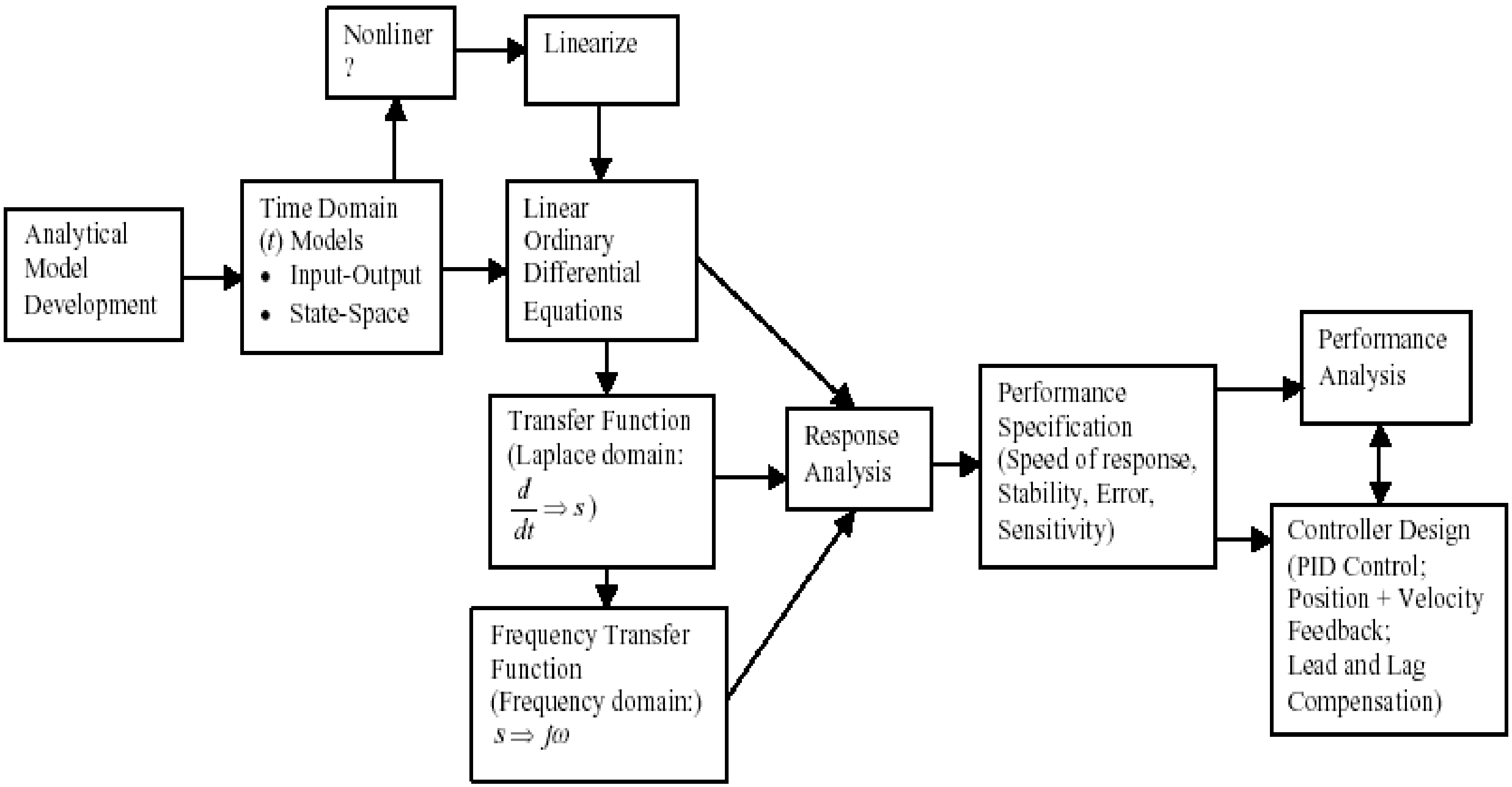
Week	Starts	Topics	Read
10	Nov 06	Root Locus Method	Section 12.5
11	Nov 13	Frequency Domain Analysis; Bode and Nyquist Diagrams; Nyquist Criterion	Sections 12.6,2.12 Appendix A
12	Nov 20	Lead and Lag Compensator Design	Section 12.8
13	Nov 27	Digital Control; Review	Section 12.10

Grade Composition

Laboratory Exercises (Five)	10%
Mid-Term Examination	20%
Final Examination	70%
Total	100%

Student Help: Office Hours: M, W, 3:00-4:00 p.m. Also, you may send me e-mail with your questions. Other face-to-face meetings can be scheduled if needed.

MECH 466 Road Map



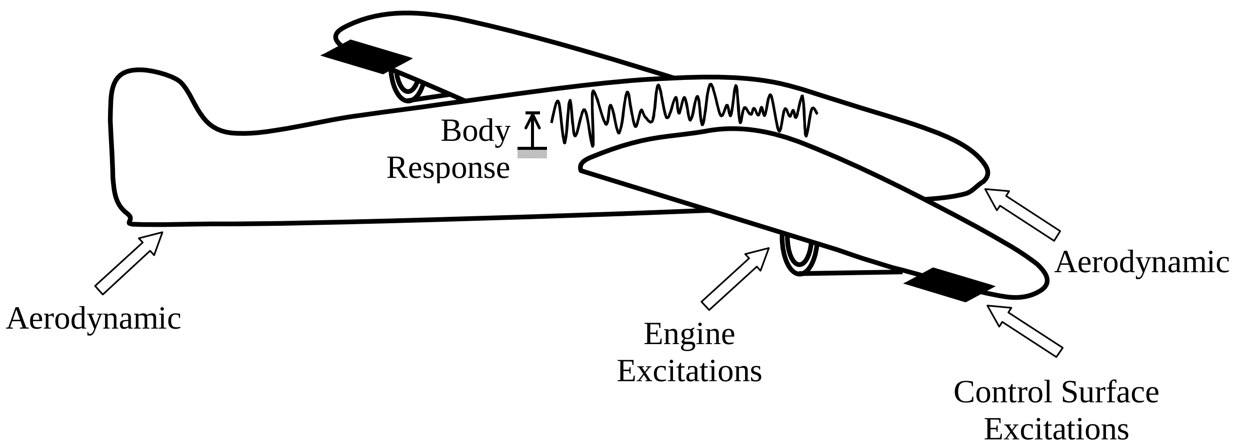
CONTROL ENGINEERING

Objective of Control:

Make a ***dynamic system*** behave in a desired manner, according some ***performance specifications***.

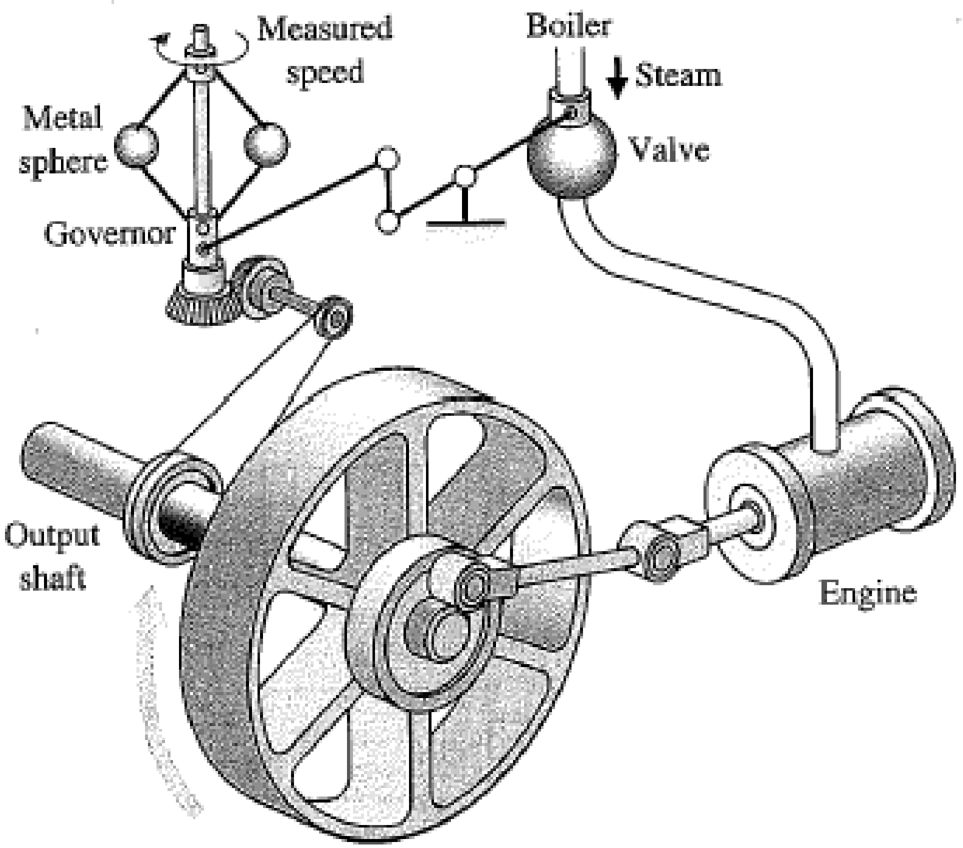
Note:

- Complex system (many inputs and many outputs, dynamic coupling, nonlinear, etc.)
- Unknown excitations
- Unknown dynamics

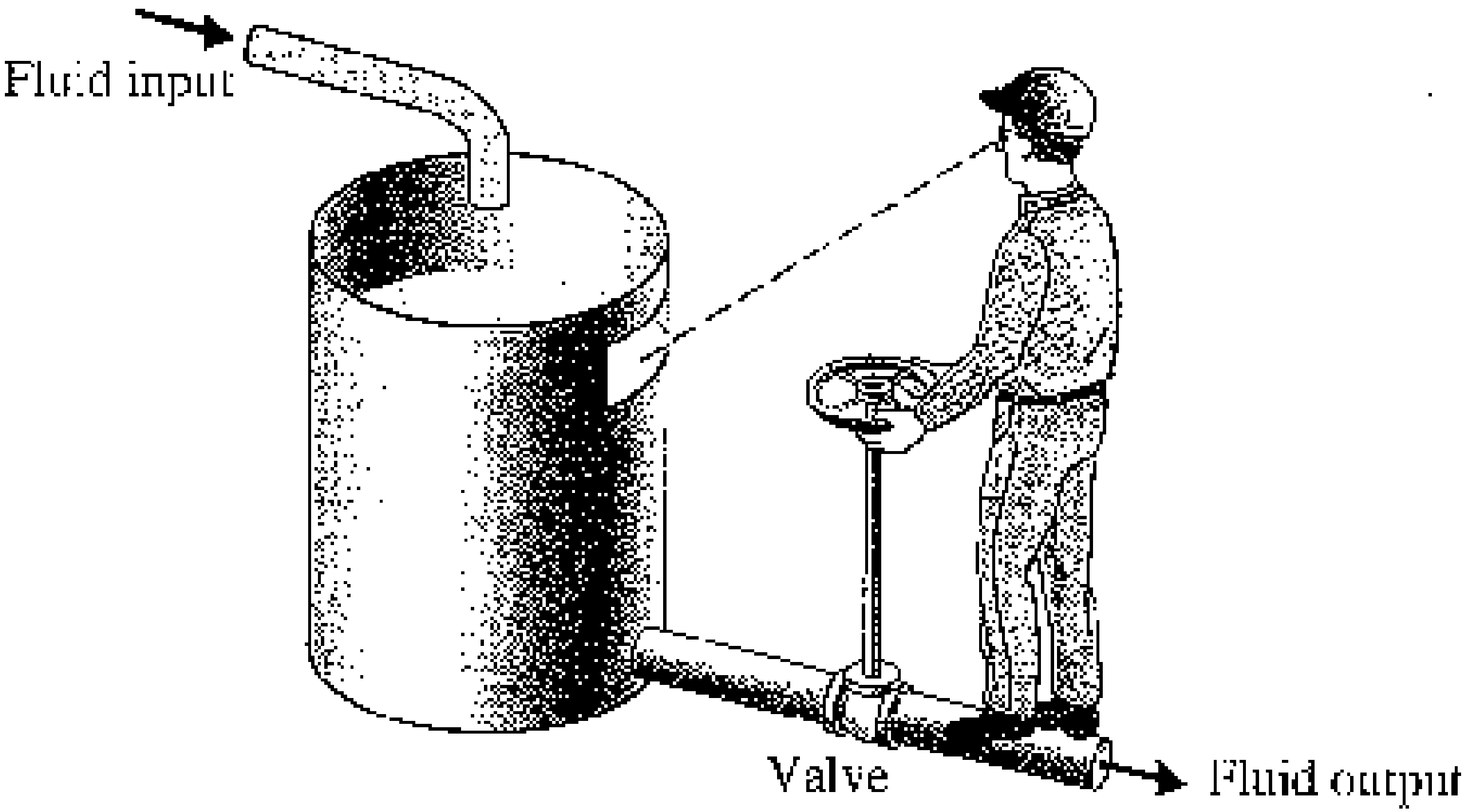


Aircraft is a Complex Control System

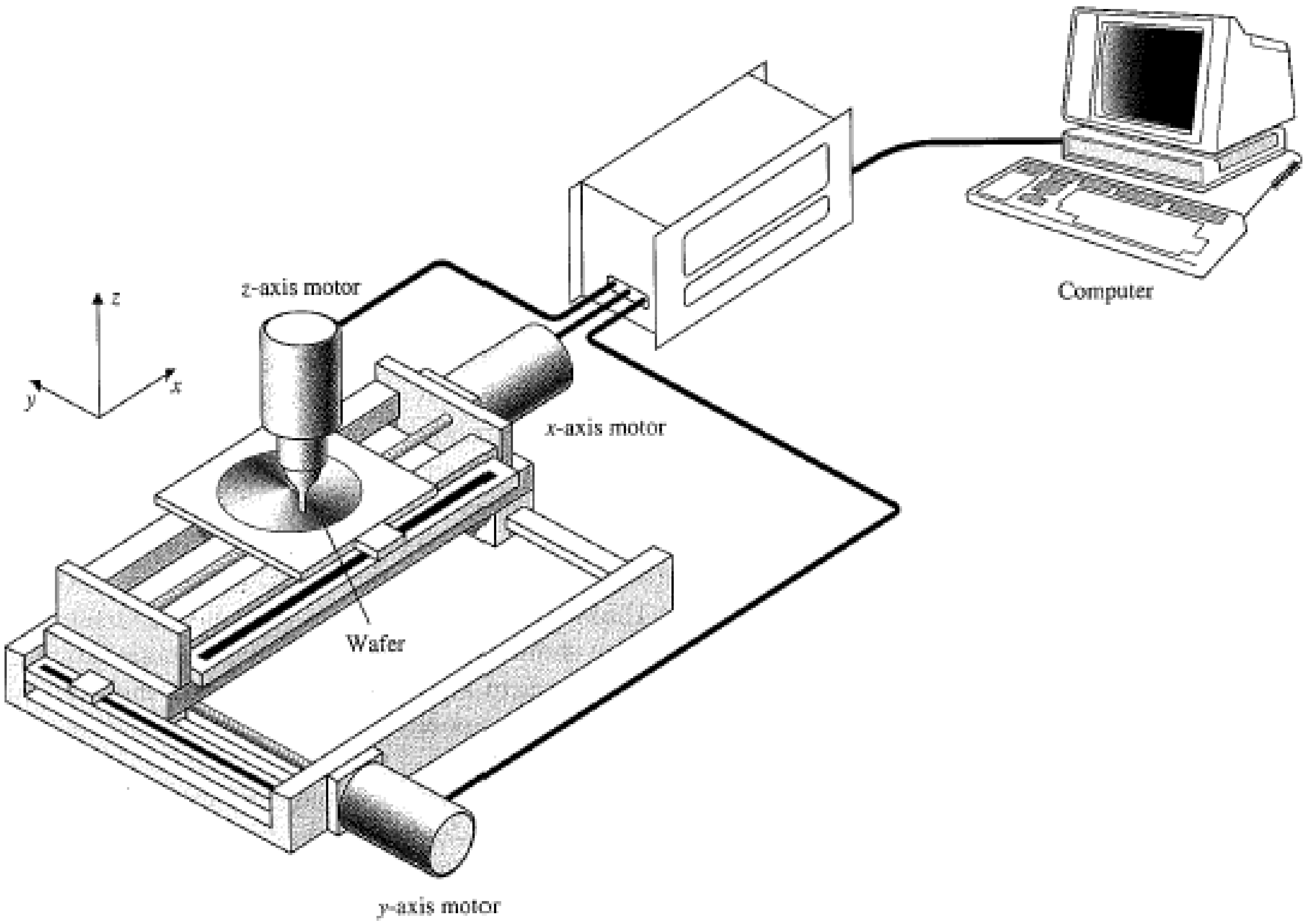
A PRIMITIVE CONTROL SYSTEM



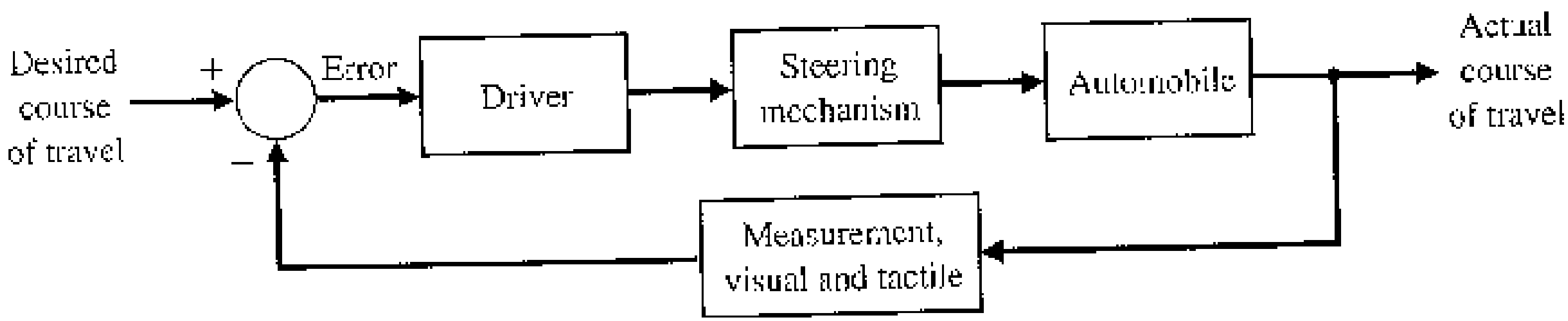
A MANUAL CONTROL SYSTEM



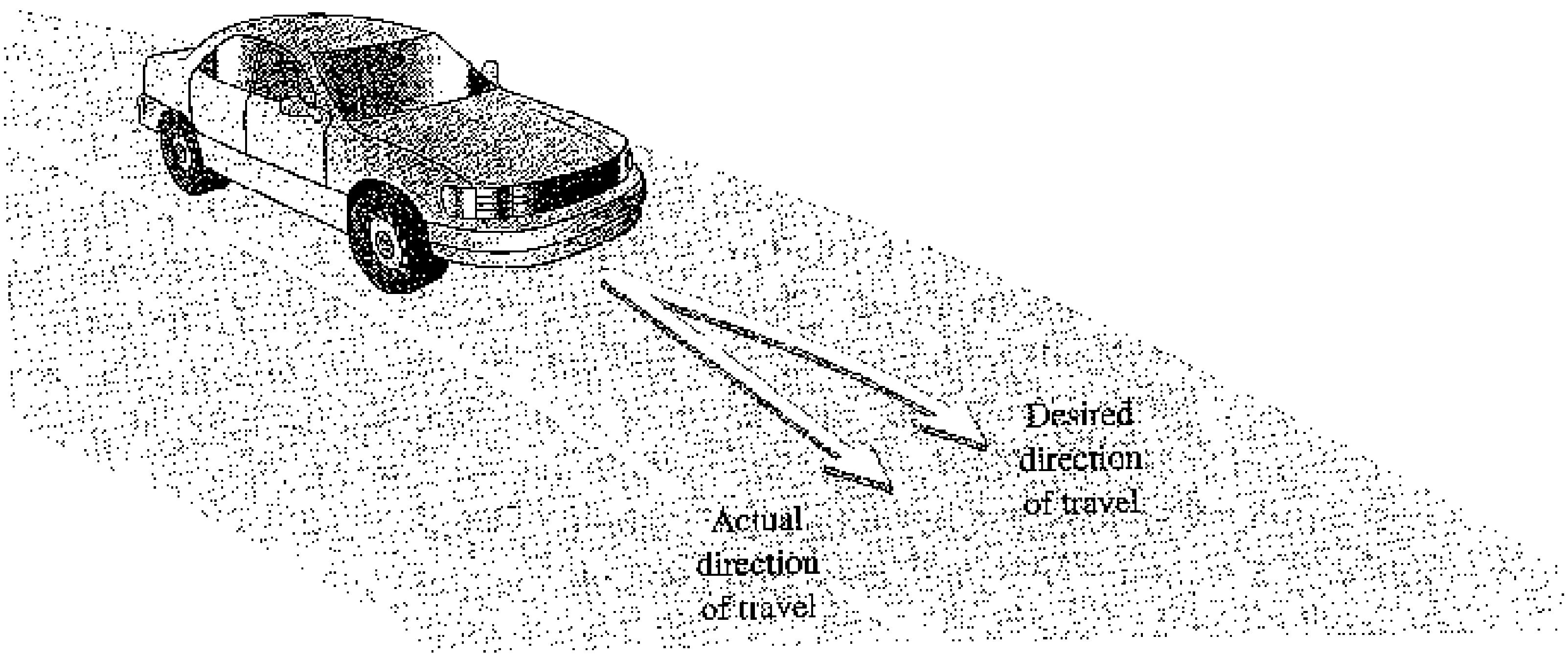
A VISION-BASED WAFER INSPECTION CONTROL SYSTEM



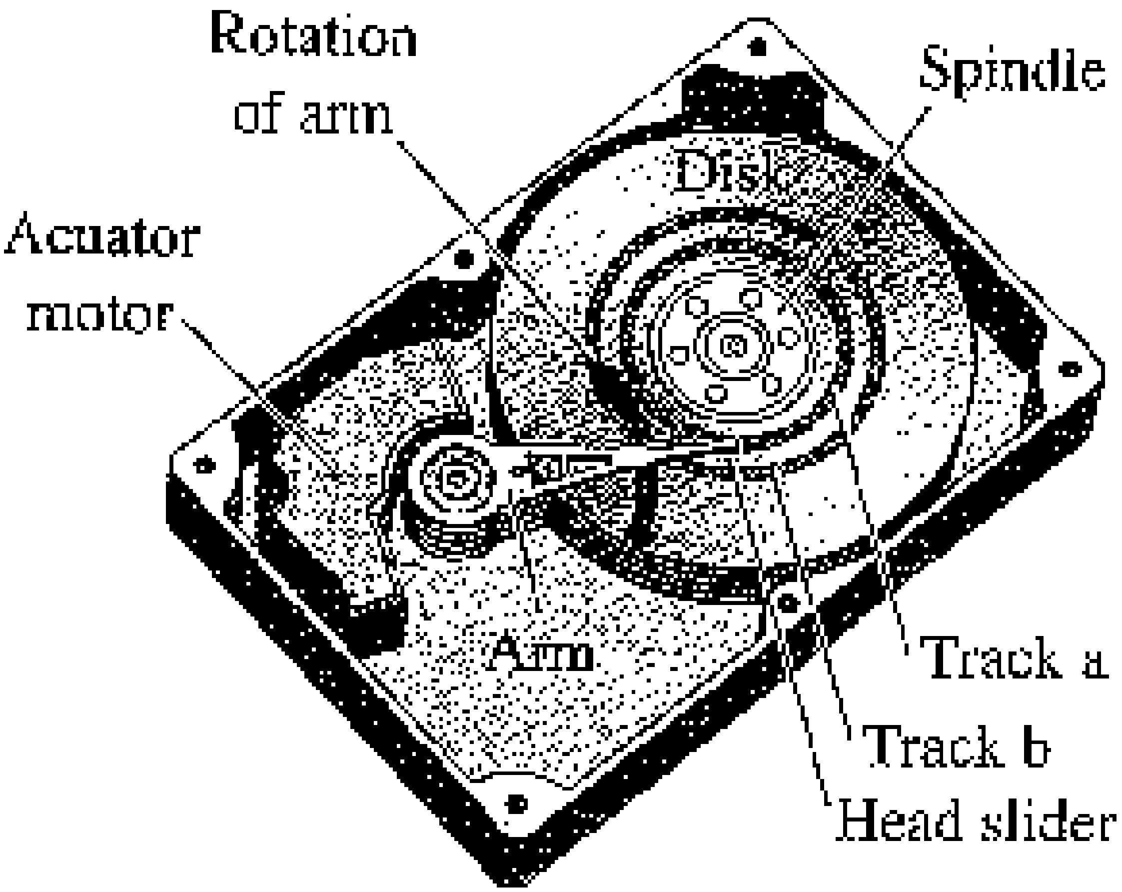
A MANUAL DRIVING CONTROL SYSTEM



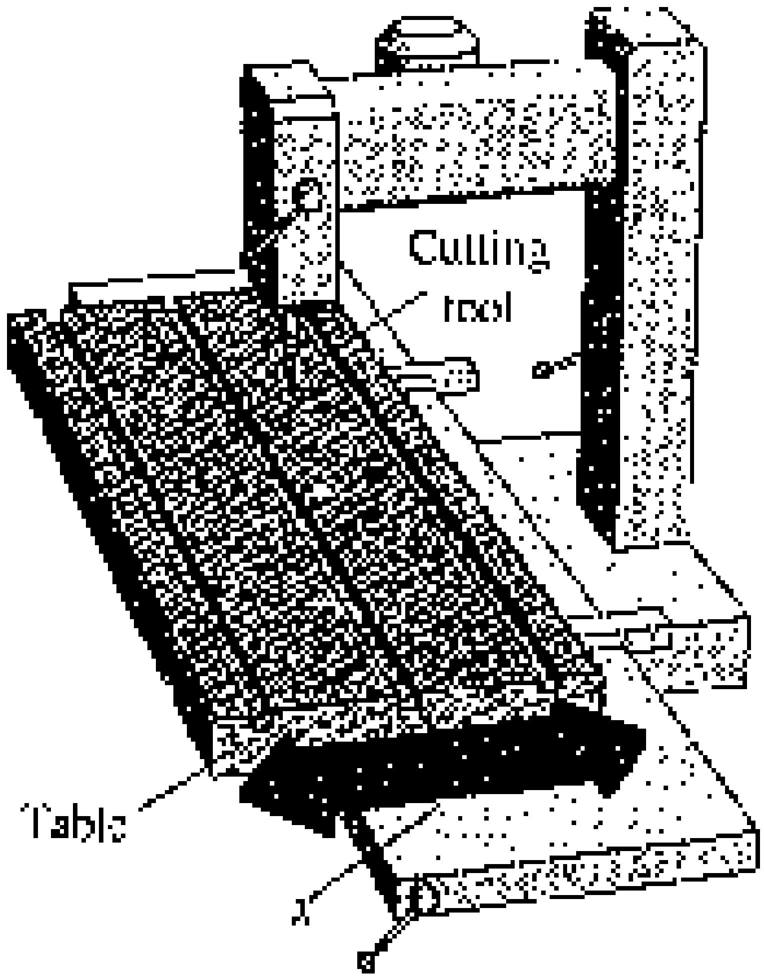
(a)



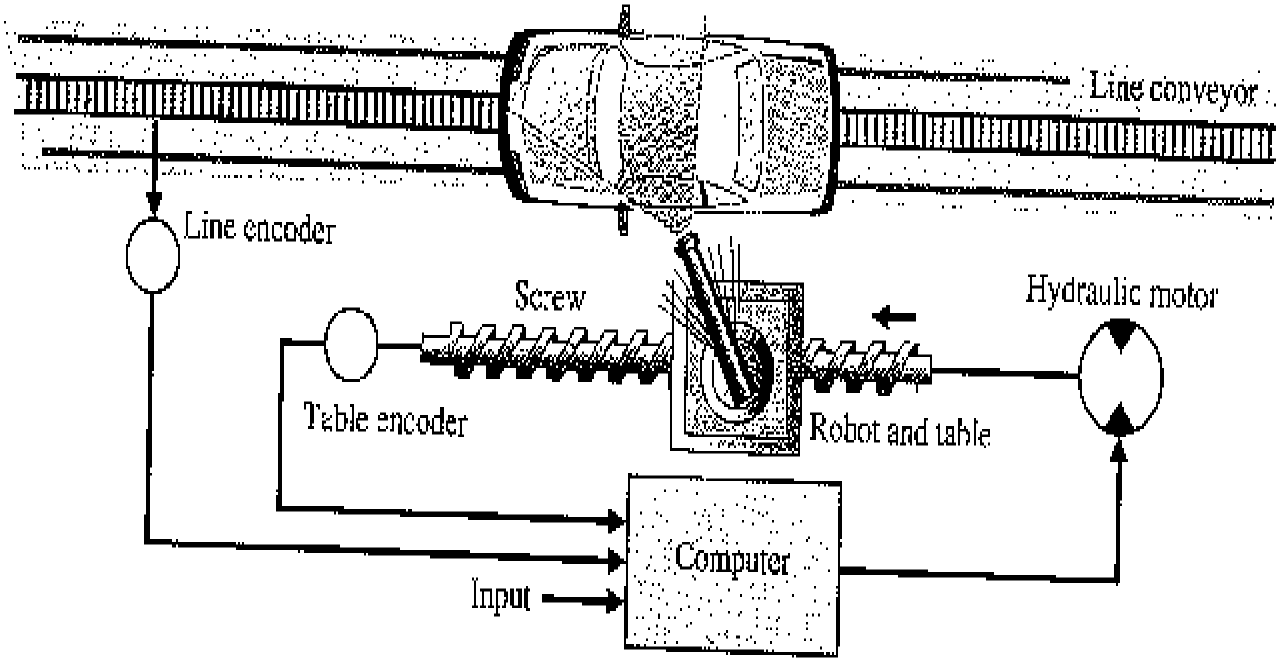
A HARD DISK DRIVE (HDD) WHICH NEEDS PRECISION CONTROL



A MACHINE TOOL WHICH NEEDS ACCURATE CONTROL



A PRIMITIVE CONTROL SYSTEM



TERMINOLOGY

Plant or Process: System to be controlled

Inputs: Excitations (known, unknown) to the system

Outputs: Responses of the system

Sensors: They measure system variables (excitations, responses, etc.)

Actuators: They drive various parts of the system.

Controller: Device that generates control signal

Control Law: Relation or scheme according to which the control signal is generated

Control System: Plant + controller, at least

(Can include sensors, signal conditioning, etc.)

Feedback Control: Control signal is determined according to plant "response"

Open-loop Control: No feedback of plant response to controller

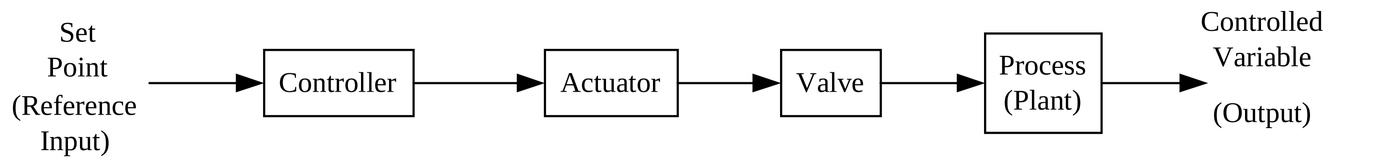
Feed-forward Control: Control signal is determined according to plant "inputs" not "outputs"

Examples of Dynamic Systems

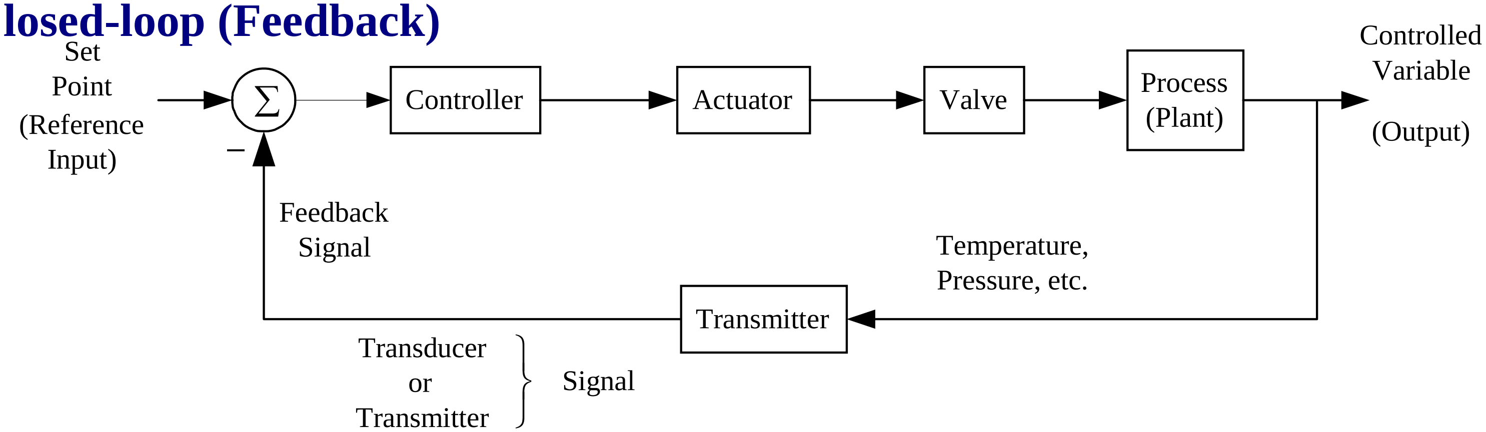
System	Typical Input	Typical Outputs
Human body	Neuroelectric pulses	Muscle contraction, body movements
Company	Information	Decisions, finished products
Power plant	Fuel rate	Electric power, pollution rate
Automobile	Steering wheel movement	Front wheel turn, direction of heading
Robot	Voltage to joint motor	Joint motions, effector motion

Basic Elements of a Process Control System

(a) Open-loop



(b) Closed-loop (Feedback)



Note: Terminology is particular to the process control practice.

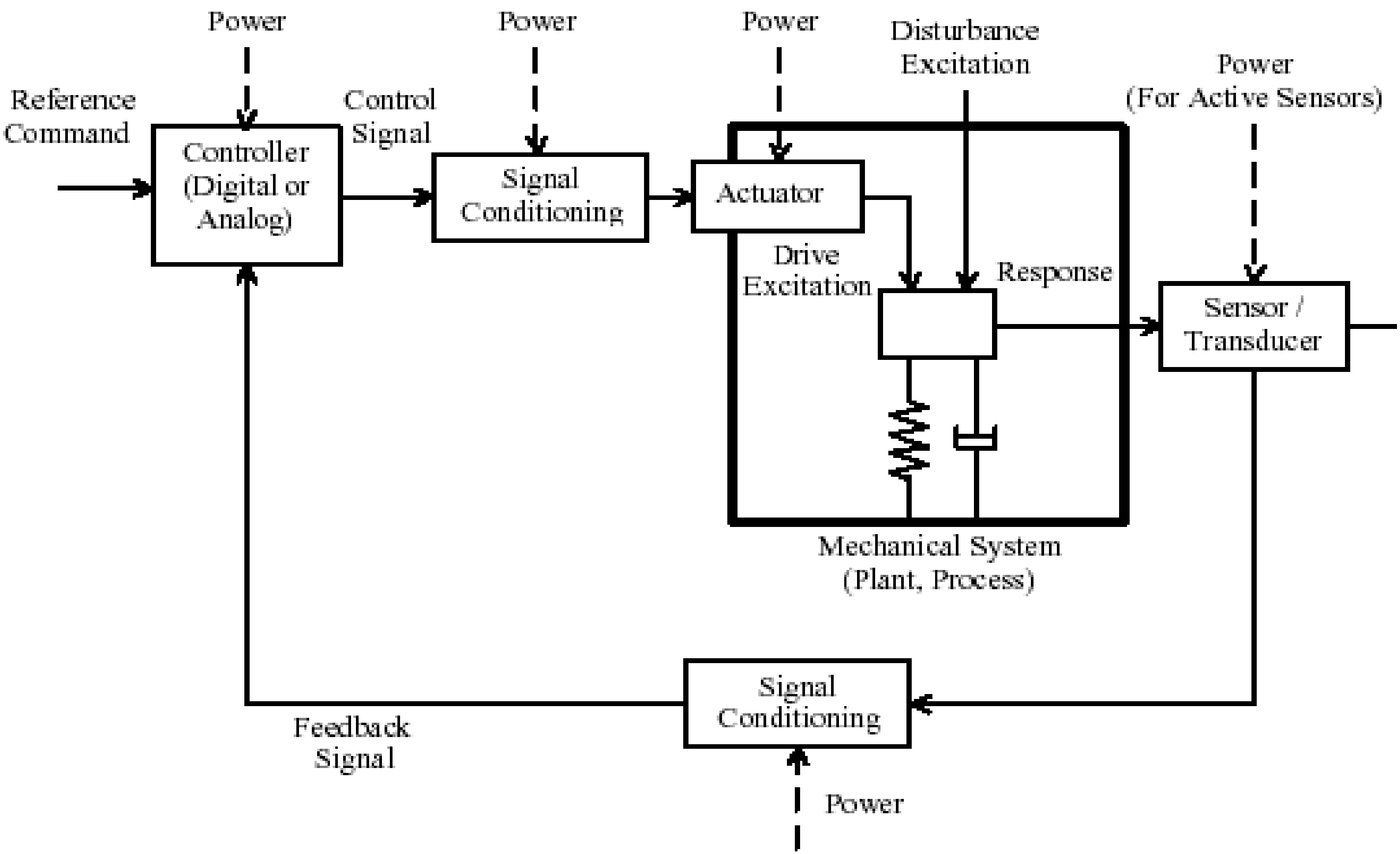
Actuator + valve => "actuator" in conventional control terminology.

Control actuators: Torque motors in servovalves, etc.

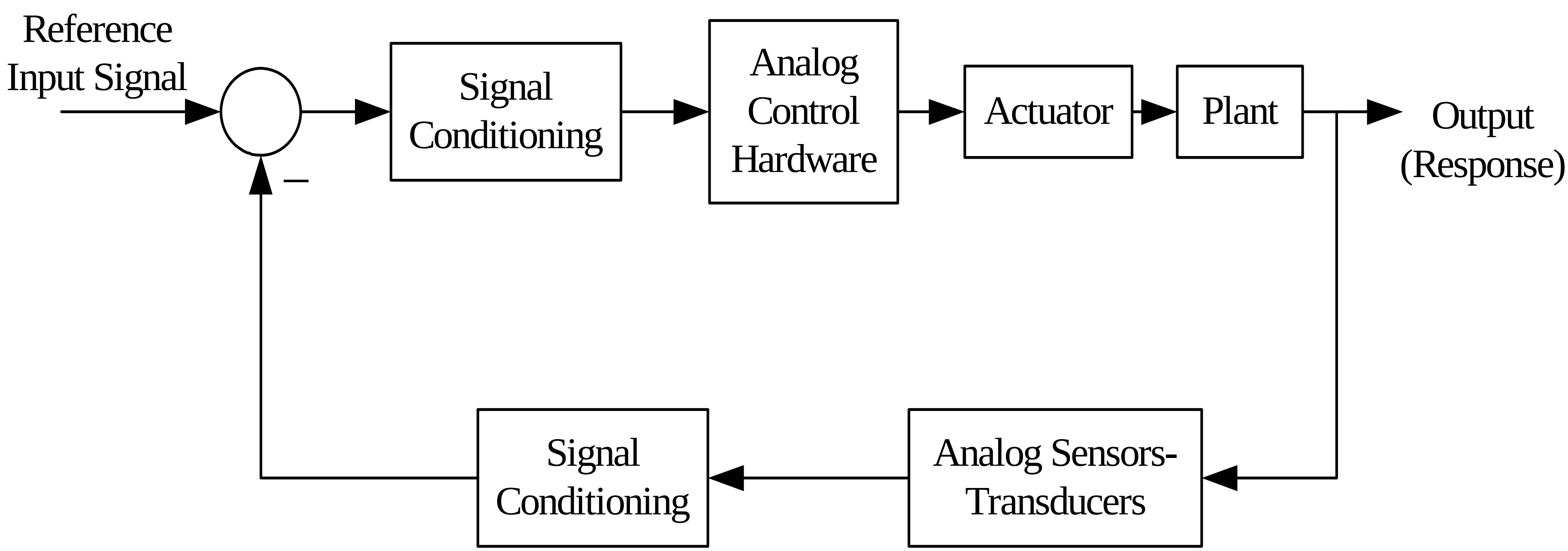
Final control element => actuator (typically a control valve).

Transmitter transmits the sensed signal to the controller (can be integrated into sensor/transducer).

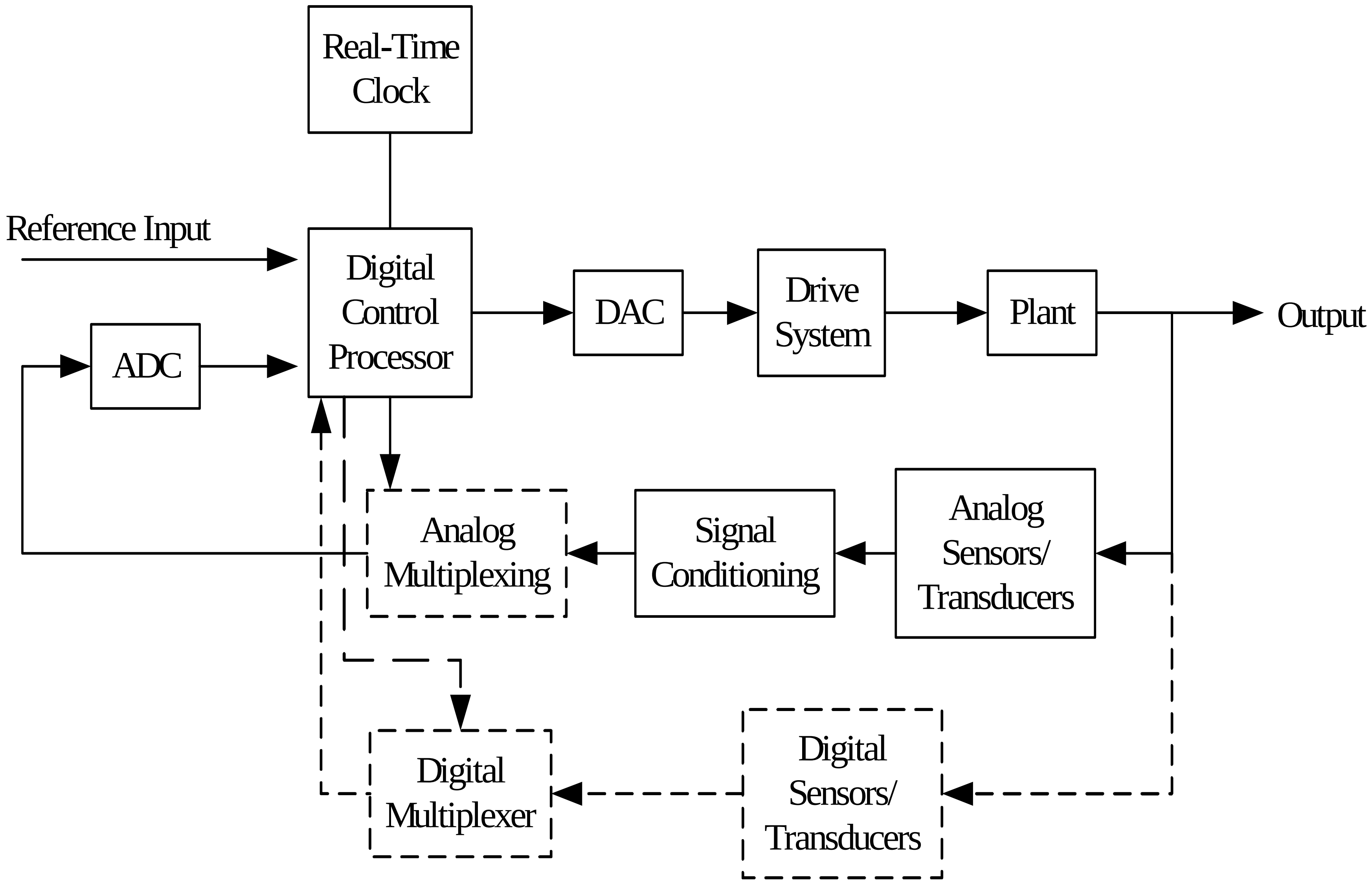
A Feedback Control System



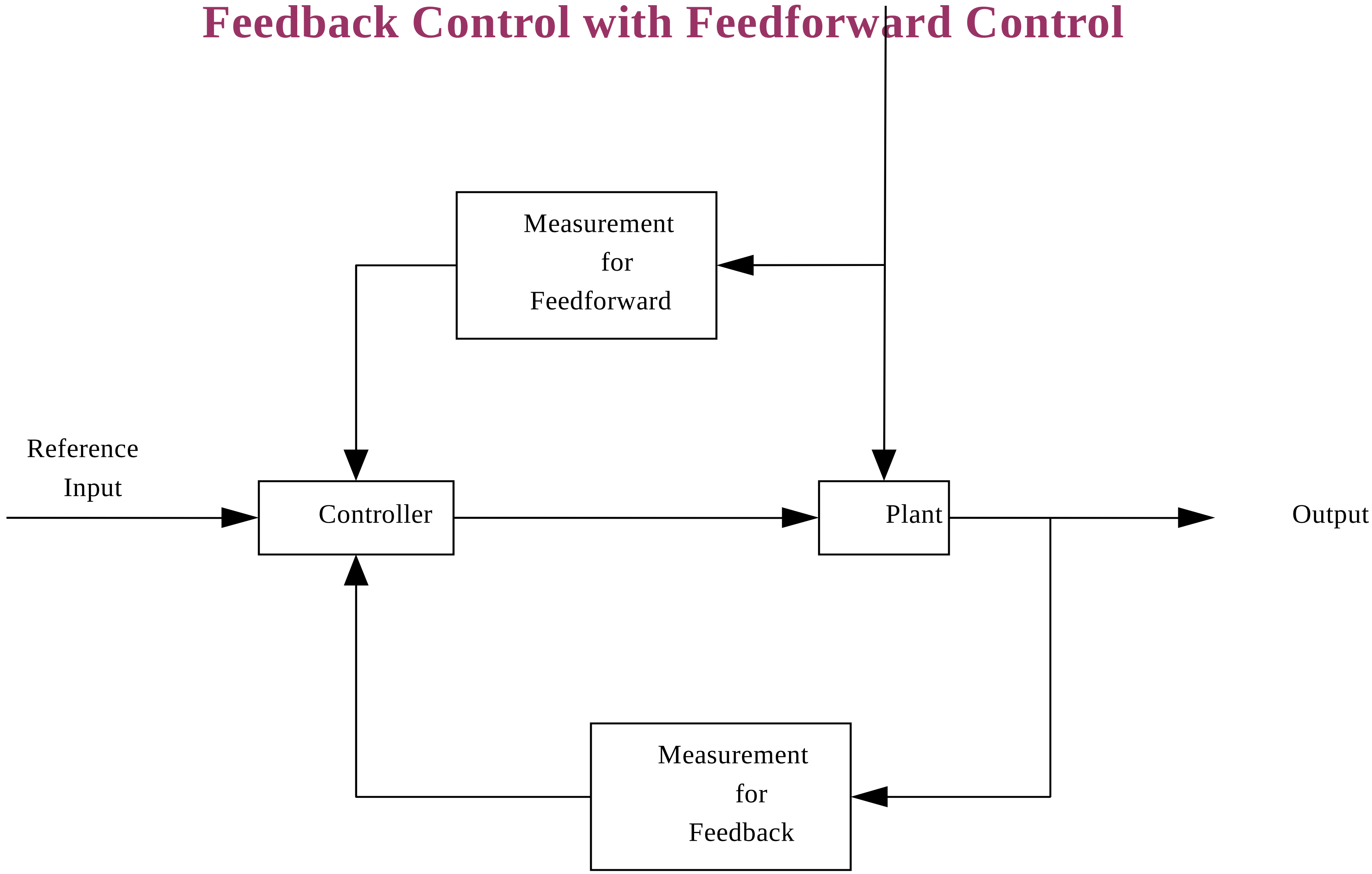
Block Diagram of an Analog Control System



Block Diagram of a Direct Digital Control System



Feedback Control with Feedforward Control



Signal Categories for Identifying Control System Types

Signal (Data) Category	Description
Analog signals (data)	Continuous in time *t*; typically represents an output of a dynamic system
Sampled data	Pulse amplitude-modulated signals Information carried by pulse amplitude Typically generated by sample-and-hold process
Digital data	Coded numerical data; the particular code (e.g., binary) determines the numerical value Typically generated by digital processors, digital transducers, and counters

Note: Sampling errors (aliasing) of sampled data and quantization errors (finite word length) of digital data.

Continuous-Time and Discrete-Time Systems

System	Analytical Model	
	Time Domain	Transfer-Function Domain
Continuous-time systems	Differential equations	Laplace transfer functions or Fourier frequency response functions
Discrete-time systems	Difference equations	Z-transform transfer functions

- **Deterministic Systems**
- **Stochastic (Random) Systems**

HISTORY OF CONTROL ENGINEERING

300 B.C. Greece (Float valves and regulators for liquid level control)
1770 James Watt (Steam engine; Governor for speed control)
1868 James Maxwell (Cambridge University, Theory of governors)
1877 E.J. Routh (Stability criterion)
1893 A.M. Lyapunov (Soviet Union, Stability theory, basis of state space formulation)
1927 H.S. Black and H.W. Bode (AT&T Bell Labs, Electronic feedback amplifier)
1930 Norbert Wiener (MIT, Theory of stochastic processes)
1932 H. Nyquist (AT&T Bell Labs, Stability criterion from Nyquist gain/phase plot)
1936 A. Callender, D.R. Hartee, and A. Porter (England, PID Control)
1948 Claude Shannon (MIT, Mathematical Theory of Communication)

HISTORY OF CONTROL ENGINEERING (Cont'd)

1948 W.R. Evans (Root locus method)

1940s Theory and applications of servomechanisms, cybernetics, and control (MIT, Bell Labs, etc.)

1959 H.M. Paynter (MIT, Bond graph techniques for system modeling)

1960s Rapid developments in State-space techniques, Optimal control, Space applications (R. Bellman and R.E. Kalman in USA, L.S. Pontryagin in USSR, NASA)

1965 Theory of fuzzy sets and fuzzy logic (L.A. Zadeh)

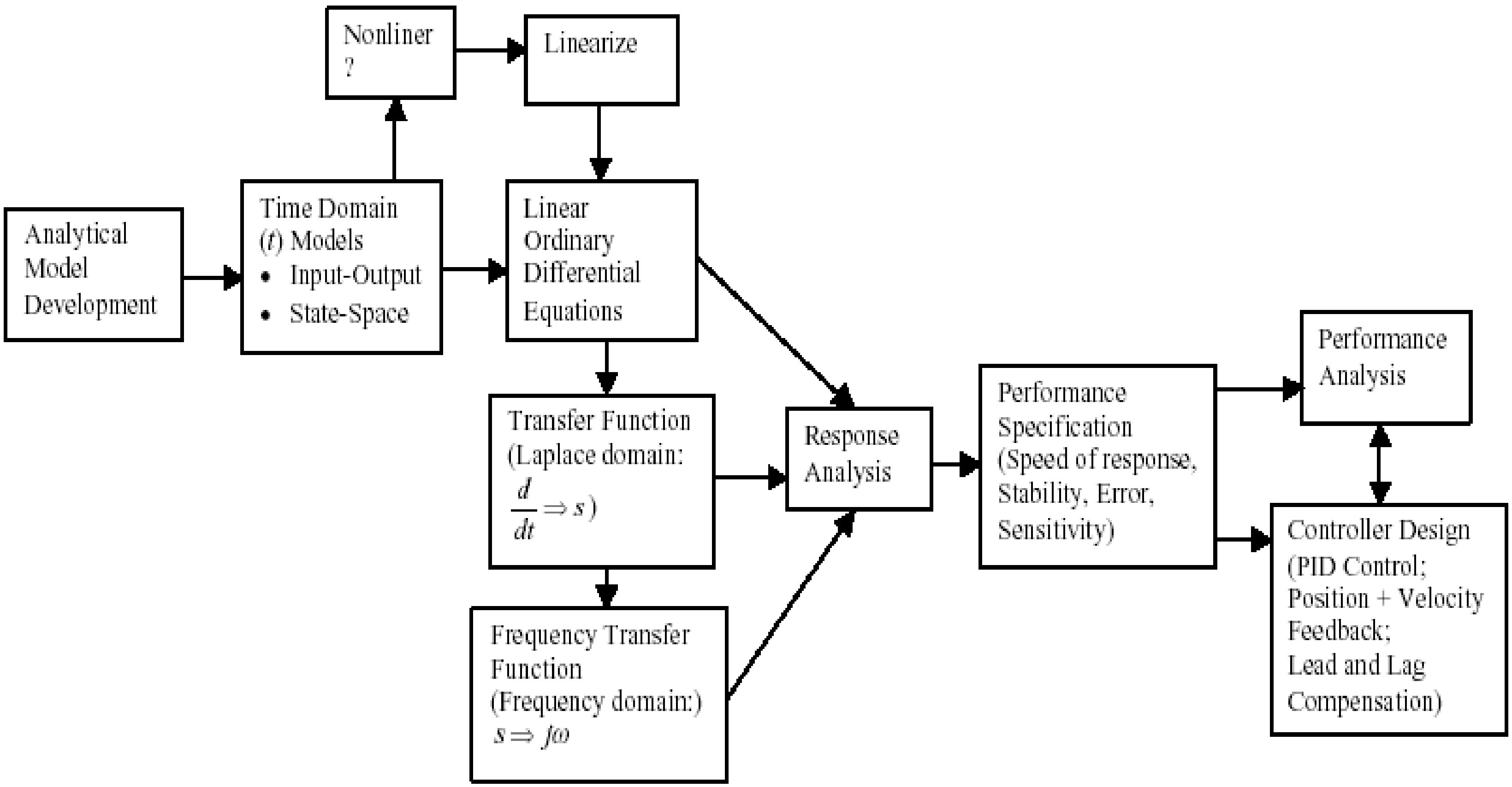
1970s Intelligent control; Developments of neural networks; Widespread developments of robotics and industrial automation (North America, Japan, Europe)

1980s Robust control; Widespread applications of robotics, and flexible automation

1990s Widespread application of smart products; Developments in Mechatronics, MEMS

Challenges: Nanotechnology, embedded, distributed, and integrated sensors, actuators, and controllers; Intelligent multiagent systems; Smart and adaptive structures; Intelligent vehicle-highway systems, etc.

MECH 466 Road Map



CONTROL TECHNIQUES

Broad Division:

1. Time domain techniques (Differential equations wrt time t; can be nonlinear)
2. Frequency domain techniques (Transfer functions—

   algebraic wrt frequency ω; typically linear)

Servo Control: To track a specified trajectory

(Commonly uses proportional-integral-derivative or PID control; typically linear)

Compensators: Hardware/software modules that help the controller to achieve the required system performance (Lead, Lag, Lead-Lag)

"Modern" Control Techniques:

(Use state-space representation; not so modern)

1. **Linear quadratic regulator (LQR): Minimize a cost function (maximize a performance index** $\Rightarrow$Optimal control)**.**
2. **Pole placement: Locate system poles (eigenvalues) to modify the modes (i.e., fundamental free natural responses) with respect to stability, speed of response, etc.** $\Rightarrow$ modal control**.**

OTHER MODERN CONTROL TECHNIQUES

1. **Linear Quadratic Gaussian (LQG) Control: LQR plus a Kalman filter. When inputs and the measurements have noise**
2. **Nonlinear Feedback Control (feedback linearization technique or FLT): Feedback signal (based on measurements or an analytical model of the plant) applied to compensate for (remove) nonlinear effects**
3. **Adaptive Control: Controller (e.g., PID) parameters are adjusted (tuned) according a performance criterion. Nonlinear.**
4. **Sliding Mode Control: Switching controller; control signal is switched between control laws to push the response towards a desired region (sliding surface). Nonlinear.**
5. **H-infinity Control: H-infinity norm (a performance criterion) is minimized. Linear.**

INTELLIGENT CONTROL

(Knowledge-based Control)

- **Knowledge base (non-analytic); e.g., a set of rules, is available regarding system behavior (through operating experience, heuristics, control expertise, etc.).**
- **Suitable for large-scale and complex systems (analytical modeling difficult; important input-output signals not available for measurement)**

Popular Approaches:

Fuzzy Logic Control

Neural-Network Control

Fuzzy-Neural Control

ANALYTICAL MODELS

Dynamic System: Response variables are functions of time, with non-negligible rates of changes.

Model: A representation of a system.

Types of Models:

1. Physical Models (Prototypes)
2. Analytical Models
3. Computer (Numerical) Models
4. Experimental Models (using input/output experimental data)

Models for Physical Dynamic Systems:

- lumped-parameter models
- continuous-parameter models.

Example: Spring element (flexibility, inertia, damping)

STEPS OF MODEL DEVELOPMENT

1. Identify system of interest (purpose, boundaries)
2. Identify/specify variables of interest (excitations, responses, etc.)
3. Approximate various segments (processes, phenomena) by ideal elements, suitably interconnected.
4. Draw a free-body diagram or circuits, with suitably isolated components.
   (a)Write **constitutive** equations (physical laws) for elements.
   (b)Write **continuity** (or conservation) equations for through variables (equilibrium of forces at joints; current balance at nodes, etc.)
   (c)Write **compatibility** equations for across (potential or path) variables (loop equations for velocities—geometric connectivity; voltages—potential balance)
   (d)Eliminate auxiliary (unwanted) variables
5. Express boundary conditions and initial conditions using system variables.

Some Linear Constitutive Relations

System Type	Constitutive Relation for Elements		
	Energy Storage		Energy Dissipating
	***A*-Type (Across) Element**	***T*-Type (Through) Element**	***D*-Type (Dissipative) Element**
Translatory Mechanical v = velocity f = force	Mass $m\frac{dv}{dt} = f$ (Newton's 2nd Law) m = mass	Spring $\frac{df}{dt} = kv$ (Hooke's Law) k = stiffness	Viscous Damper $f = bv$ b = damping constant
Electrical v = voltage i = current	Capacitor $C\frac{dv}{dt} = i$ C = capacitance	Inductor $L\frac{di}{dt} = v$ L = inductance	Resistor $Ri = v$ R = resistance
Thermal T = temperature difference Q = heat transfer rate	Thermal Capacitor $C_t\frac{dT}{dt} = Q$ C_t = thermal capacitance	None	Thermal Resistor $R_t Q = T$ R_t = thermal resistance
Fluid P = pressure difference Q = volume flow rate	Fluid Capacitor $C_f\frac{dP}{dt} = Q$ C_f = fluid capacitance	Fluid Inertor $I_f\frac{dQ}{dt} = P$ I_f = inertance	Fluid Resistor $R_f Q = P$ R_f = fluid resistance

Force-Current Analogy

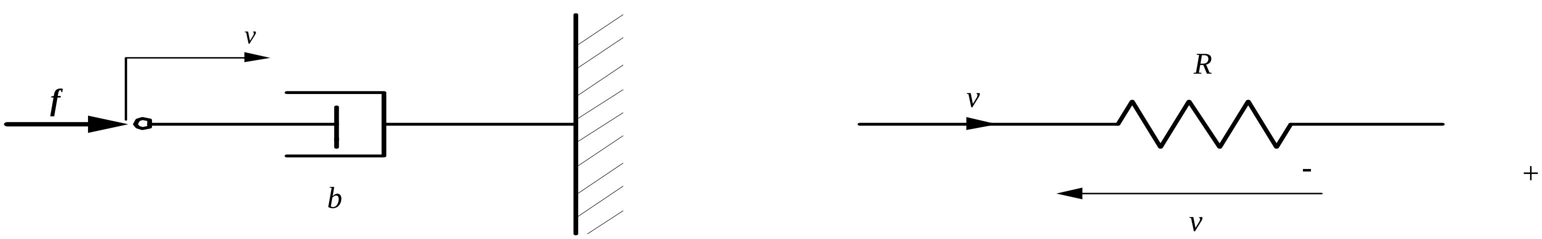
System Type	Mechanical	Electrical
System-response Variables: **Through Variables** **Across Variables**	 Force f Velocity v	 Current i Voltage v
System Parameters	M K B	C $1/L$ $1/R$

Notes for Inertia Element:

1. Velocity can represent the state of an inertia element. (Velocity at any time t completely determined from initial velocity and applied force; Energy of an inertia element can be represented by v alone.
2. Velocity across an inertia element cannot change instantaneously unless an infinite force/torque is applied.
3. A finite force cannot cause an infinite acceleration. A finite instantaneous change (step) in velocity will require an infinite force. Hence, for an inertia element, v is a natural output (or state) and f is a natural input.
4. **Note:** Similarly, voltage is state variable for a capacitor.

A Mass Element and an Analogous Capacitor Element.

A Spring Element and an Analogous Inductor Element.



$f = bv$

$i = \frac{1}{R} v$

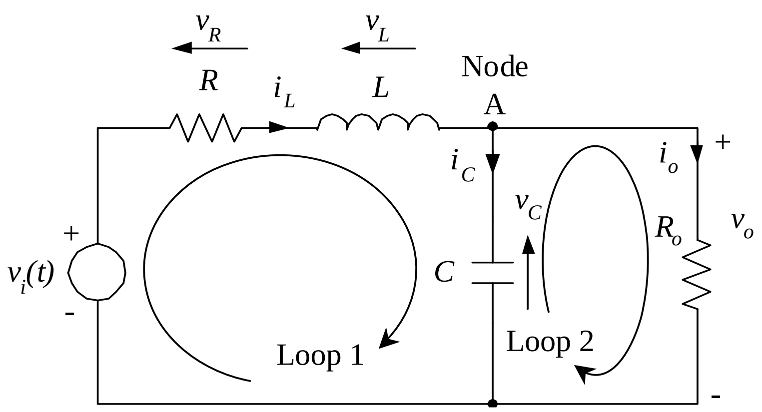
Viscous Damper and Analogous Electrical Resistor.

Natural Oscillations: One type of stored energy is converted to another type repeatedly, back and forth.

System Order: Minimum number of state variables (or, initial conditions) needed to represent/solve the dynamic system; typically, number of independent energy storage elements

Example (Electrical)

The circuit shown in the figure consists of an inductor L, a capacitor C, and two resistors R and R_o. The input is the voltage $v_i(t)$ and the output is the voltage v_o across the resistor R_o.



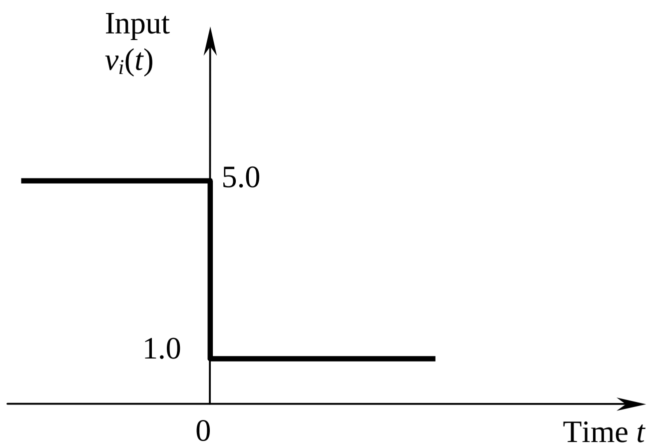
(a) Obtain a complete state-space model for the system.

(b) Obtain an input-output differential equation for the system.

Example (Cont'd)

(c) Obtain expressions for undamped natural frequency and the damping ratio of the system.

(d) The system starts at steady state with an input of 5 V (for all $t < 0$). Then suddenly, the input is dropped to 1 V (for all $t > 0$), which corresponds to a step input as shown below.



For $R = R_o = 1\ \Omega$, $L = 1$ H, and $C = 1$ F, what are the initial conditions of the system and their derivatives at both $t = 0^-$ and $t = 0^+$? What are the final (steady state) values of the state variables and the output variable? Sketch the nature of the system response.

Solution

(a)

State Variables:

Current through independent inductors (i_L)
Voltage across independent capacitors (v_c)

Constitutive Equations:

$$v_L = L\frac{di_L}{dt}$$

$$i_C = C\frac{dv_C}{dt}$$

$$v_R = Ri_L \ ; \qquad v_o = Ri_o$$

First two equations are for independent energy storage elements ➔ State-space shell.

Continuity Equation:

Node A (Kirchhoff's Current Law): $i_L - i_C - i_o = 0$

Compatibility Equations:

Loop 1 (Kirchhoff's Voltage Law): $v_i - v_R - v_L - v_C = 0$

Loop 2 (Kirchhoff's Voltage Law): $v_C - v_o = 0$

Eliminate auxiliary variables:

$$L\frac{di_L}{dt} = v_L = v_i - v_R - v_C = v_i - Ri_L - v_C$$

$$C\frac{dv_C}{dt} = i_C = i_L - i_o$$

$$= i_L - \frac{v_o}{R_o} = i_L - \frac{v_C}{R_o}$$

==> State equations:

$$\frac{di_L}{dt} = \frac{1}{L}[-Ri_L - v_C + v_i] \qquad \text{(i)}$$

$$\frac{dv_C}{dt} = \frac{1}{C}[i_L - \frac{v_C}{R_o}] \qquad \text{(ii)}$$

Output Equation:

$$v_o = v_c$$

Vector-Matrix Representation:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\boldsymbol{y} = \boldsymbol{C}\,\boldsymbol{x}$$

where:

System matrix $\boldsymbol{A} = \begin{bmatrix} -R/L & -1/L \\ 1/C & -1/(R_o C) \end{bmatrix}$

Input gain matrix $\boldsymbol{B} = \begin{bmatrix} 1/L \\ 0 \end{bmatrix}$

Measurement gain matrix $\boldsymbol{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$

State vector = $\boldsymbol{x} = \begin{bmatrix} i_L \\ v_C \end{bmatrix}$

Input = $\boldsymbol{u} = [v_i]$

Output = $\boldsymbol{y} = [v_o]$

(b)
From (ii):

$$i_L = C\frac{dv_C}{dt} + \frac{v_C}{R_o}$$

Substitute in (i) for i_L:

$$L\frac{d}{dt}(C\frac{dv_C}{dt} + \frac{v_C}{R_o}) = -R(C\frac{dv_C}{dt} + \frac{v_C}{R_o}) - v_C + v_i$$

This simplifies to the **input-output differential equation** (since $v_o = v_C$)

$$LC\frac{d^2v_o}{dt^2} + (\frac{L}{R_o} + RC)\frac{dv_o}{dt} + (\frac{R}{R_o} + 1)v_o = v_i \qquad \text{(iii)}$$

(c)

The input-output differential equation is of the form

$$\frac{d^2 v_o}{dt^2} + 2\zeta\omega_n \frac{dv_o}{dt} + \omega_n{}^2 v_o = \frac{1}{LC} v_i$$

Hence,

Natural frequency $\omega_n = \sqrt{\frac{1}{LC}\left(\frac{R}{R_o}+1\right)}$ (iv)

Damping ratio $\zeta = \frac{1}{2\sqrt{LC\left(\frac{R}{R_o}+1\right)}}\left(\frac{L}{R_o}+RC\right)$ (v)

Note: $\frac{1}{LC}$ has units of (frequency)^2.

RC and $\frac{L}{R_o}$ have units of "time" (i.e., time constant).

(d)

Initial Conditions:

For $t < 0$ (initial steady state):

$$\frac{di_L}{dt} = 0$$

$$\frac{dv_c}{dt} = 0$$

➔

(i): $$\frac{di_L(0^-)}{dt} = 0 = \frac{1}{L}[-Ri_L(0^-) - v_C(0^-) + v_i(0^-)]$$

(ii): $$\frac{dv_C(0^-)}{dt} = 0 = \frac{1}{C}[i_L(0^-) - \frac{v_C(0^-)}{R_o}]$$

Substitute the given parameter values $R = R_o = 1\ \Omega$, $L = 1$ H, and $C = 1$ F, and the input $v_i(0^-) = 5.0$:

$$-i_L(0^-) - v_C(0^-) + 5 = 0$$

$$i_L(0^-) - v_C(0^-) = 0$$

➔ $$i_L\left(0^-\right) = 2.5\ \text{A},\quad v_c\left(0^-\right) = 2.5\ \text{V}$$

Note: State variables cannot undergo step changes (because that violates the corresponding physical laws – constitutive equations). Specifically, Inductor cannot have a step change in current (needs infinite voltage). Capacitor cannot have a step change in voltage (needs infinite current). Hence,

$$i_L\left(0^+\right) = i_L\left(0^-\right) = 2.5 \text{ A}$$

$$v_c\left(0^+\right) = v_c\left(0^-\right) = 2.5 \text{ V}$$

Note: Since $v_i(0^+) = 1.0$

(i): $$\frac{di_L(0^+)}{dt} = -i_L(0^+) - v_C(0^+) + 1.0 = -2.5 - 2.5 + 1.0 = -4.0 \text{ A/s} \neq 0$$

(ii): $$\frac{dv_C(0^+)}{dt} = i_L(0^+) - v_C(0^+) = 2.5 - 2.5 = 0.0 \text{ V/s}$$

Final Values: As t $\rightarrow \infty$ (at final steady state)

$$\frac{di_L}{dt} = 0$$

$$\frac{dv_c}{dt} = 0$$

and $v_i = 1.0$

Substitute:

(i): $\frac{di_L(\infty)}{dt} = 0 = -i_L(\infty) - v_C(\infty) + 1.0$

(ii): $\frac{dv_C(\infty)}{dt} = 0 = i_L(\infty) - v_C(\infty)$

Solution: $i_L(\infty) = 0.5 \text{ A}$, $v_c(\infty) = 0.5 \text{ V}$

For the given parameter values,

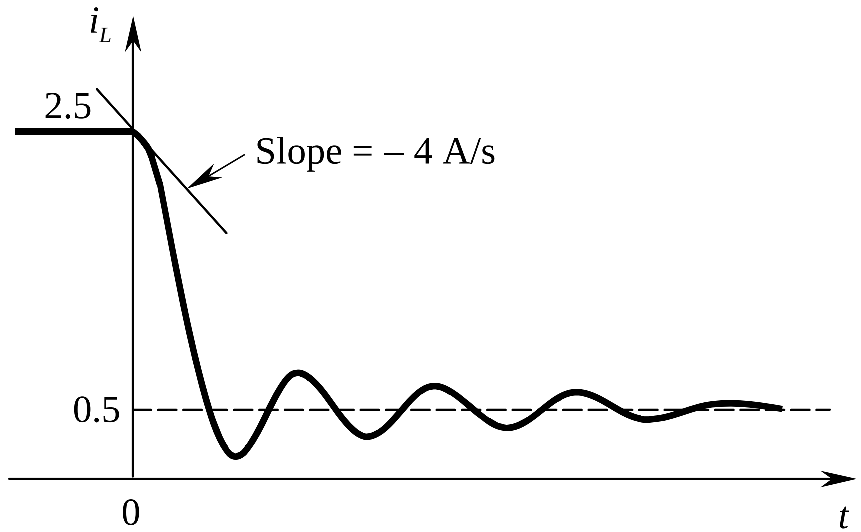
(iii): $\frac{d^2 v_o}{dt^2} + 2\frac{dv_o}{dt} + 2v_o = 1$

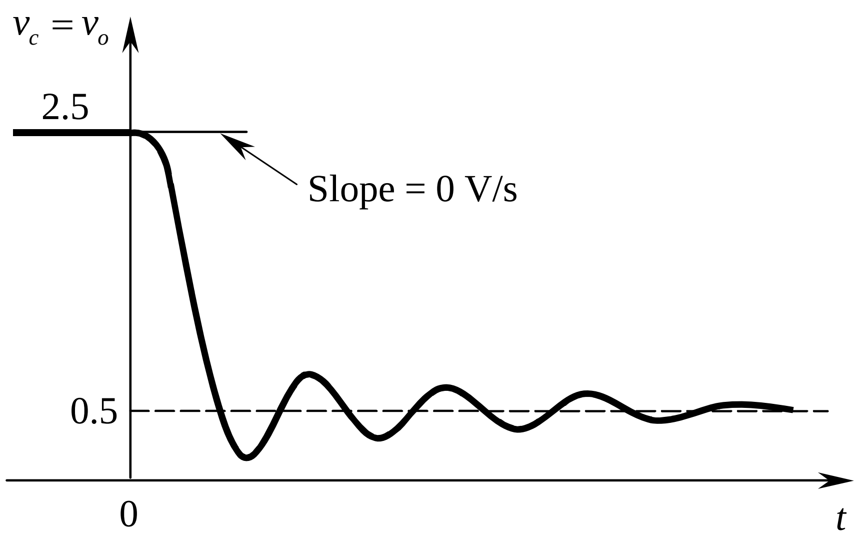
Hence, $\omega_n = \sqrt{2}$ and $2\zeta\omega_n = 2$, or, $\zeta = 1/\sqrt{2}$

➔ an underdamped system, producing an oscillatory response

Note: Output $v_o = v_c$

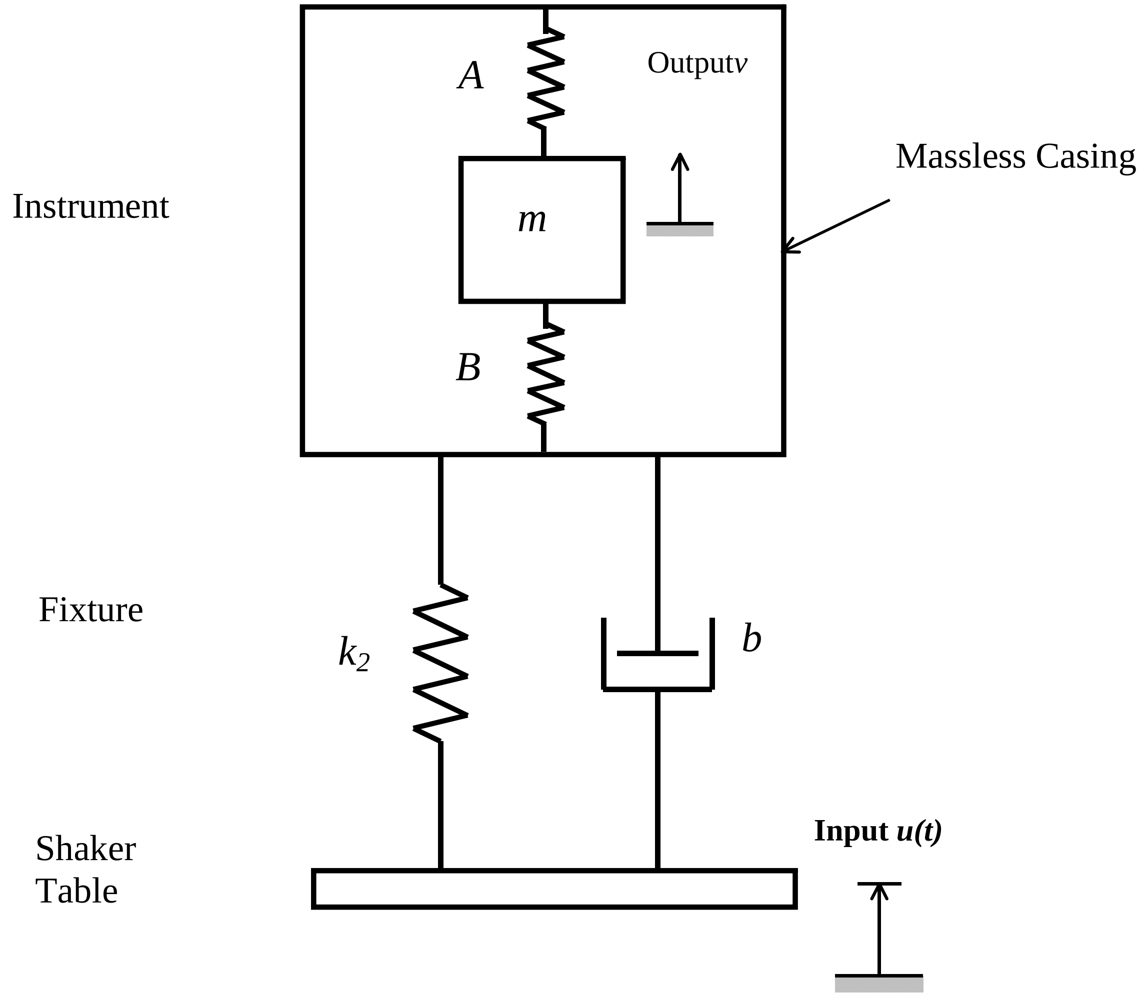
Response



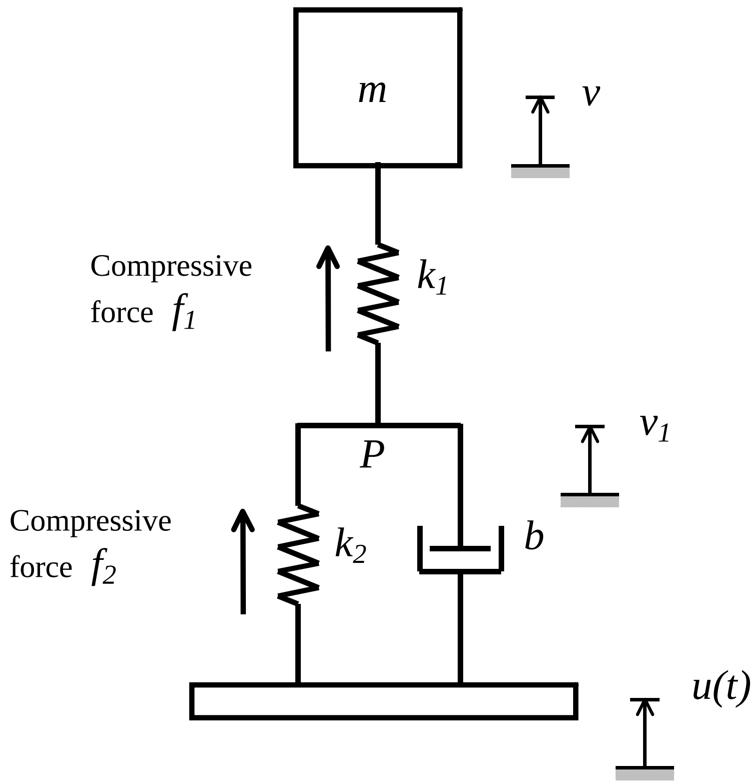


<u>Example (Mechanical)</u>: Shaker Test Setup for an Instrument

Supporting platform (shaker table) is movable, with velocity *u(t)*.



An Equivalent Model



State Variables:

1. Force through independent springs.
2. Velocity across independent masses.

Constitutive Equations:

For mass: $f_1 = m\dot{v}$

For spring 1: $\dot{f}_1 = k_1(v_1 - v\ )$

For spring 2: $\dot{f}_2 = k_2(u - v_1)$

For damper: $f_1 - f_2 = b(u - v_1)$

State-space Shell: First three equations
Auxiliary Equation: Fourth equation

Eliminate auxiliary variable v_1 $\Rightarrow$ State Equations:

$$\dot{v}\ = \frac{1}{m} f_1$$

$$\dot{f}_1 = -k_1 v\ - \frac{k_1}{b} f_1 + \frac{k_1}{b} f_2 + k_1 u$$

$$\dot{f}_2 = \frac{k_2}{b} f_1 - \frac{k_2}{b} f_2$$

Output Equation:

$$v = v$$

Vector-Matrix Form:

State vector $\boldsymbol{x} = \begin{bmatrix} v \\ f_1 \\ f_2 \end{bmatrix}$ (3^{rd} order)

Input vector $\boldsymbol{u} = [u(t)]$ (a scalar)

Output vector $\boldsymbol{y} = [v]$ (a scalar)

$$\boldsymbol{A} = \begin{bmatrix} 0 & \frac{1}{m} & 0 \\ -k_1 & -\frac{k_1}{b} & \frac{k_1}{b} \\ 0 & \frac{k_2}{b} & -\frac{k_2}{b} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ k_1 \\ 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Fluid Elements

System (State) Variables:

Pressure (across variable) of each independent fluid capacitor (*A*-type element)

Volume flow rate (through variable) of each independent fluid inertor (*T*-type element)

Fluid Capacitor

$$C_f \frac{dP}{dt} = Q$$

Note: Fluid capacitor stores potential energy (a "fluid spring")
Mechanical *A*-type element (inertia) stores kinetic energy.

For a liquid control volume V of bulk modulus β: $C_{bulk} = \frac{V}{\beta}$

For an isothermal (constant temperature, slow-process) gas of volume V and pressure P: $C_{comp} = \frac{V}{P}$

Fluid Capacitor (Cont'd)

For an adiabatic (zero heat transfer, fast-process) gas: $C_{comp} = \frac{V}{kP}$

Ratio of specific heats at constant pressure and constant volume $k = \frac{c_p}{c_v}$

For an incompressible fluid contained in a flexible vessel of area *A* and stiffness *k*:

$$C_{elastic} = \frac{A^2}{k}$$

For a fluid with bulk modulus in a flexible container,

Equivalent capacitance = $C_{bulk} + C_{elastic}$.

For an incompressible fluid column (area of cross-section *A* and density ρ):

$$C_{grav} = \frac{A}{\rho g}$$

Fluid Inertor

$$I_f \frac{dQ}{dt} = P$$

An intertor is a *T*-type element, and stores kinetic energy
Note: mechanical *T*-type element (spring) stores potential energy.

For a flow with uniform velocity distribution across an area *A* and over a length segment Δx

$$I_f = \rho \frac{\Delta x}{A}$$

For a non-uniform velocity distribution

$$I_f = \alpha\rho \frac{\Delta x}{A}$$

(Correction factor = α. For a flow of circular cross-section with a parabolic velocity distribution, $\alpha = 2.0$)

Fluid Resistor

In the approximate, linear case: $P = R_f Q$

In the general, nonlinear case: $P = K_R Q^n$

For viscous flow through a uniform pipe with circular cross-section of diameter *d* and length *l*:

$$R_f = 128\,\mu l \frac{\Delta x}{\pi d^4}$$

For a rectangular cross-section of height *b* which is much smaller than its width *w*:

$$R_f = 12 \mu l \frac{\Delta x}{w b^3}$$

$$\mu = \upsilon \rho$$

(μ = absolute viscosity (or, dynamic viscosity); υ = kinematic viscosity)

Valve, Orifice, Constriction of area *A* (Nonlinear Resistors): $Q = A c_d \sqrt{\frac{\Delta P}{\rho}}$

c_d = discharge coefficient

Useful Laplace Transform Pairs

$\mathcal{L}^{-1} F(s) = f(t)$	$\mathcal{L}\, f(t) = F(s)$
$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\exp(st)ds$	$\int_0^\infty f(t)\exp(-st)\, dt$
$k_1 f_1(t) + k_2 f_2(t)$	$k_1 F_1(s) + k_2 F_2(s)$
$\exp(-at) f(t)$	$F(s+a)$
$f(t-\tau)$	$\exp(-\tau s)F(s)$
$f^{(n)}(t) = \frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1} f(0^+) - s^{n-2} f^1(0^+) - \cdots - f^{n-1}(0^+)$
$\int_{-\infty}^{t} f(t)dt$	$\frac{F(s)}{s} + \frac{\int_{-\infty}^{0} f(t)dt}{s}$
Impulse function $\delta(t)$	1
Step function $\mathcal{U}(t)$	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
$\exp(-at)$	$\frac{1}{s+a}$
sin $\omega_n t$	$\frac{\omega_n}{s^2+\omega_n{}^2}$
os $\omega_n t$	$\frac{s}{s^2+\omega_n{}^2}$